Asymptotic Confidence Bands for the Estimated Autocovariance and Autocorrelation Functions of Vector Autoregressive Models

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Abstract

This paper provides closed-form formulae for computing the asymptotic covariance matrices of the estimated autocovariance and autocorrelation functions of stable VAR models by means of the delta method. These covariance matrices can be used to construct asymptotic confidence bands for the estimated autocovariance and autocorrelation functions to assess the underlying estimation uncertainty. The usefulness of the formulae for empirical work is illustrated by an application to inflation and output gap data for the U.S. economy indicating the existence of a significant short-run Phillips-curve tradeoff.

Keywords: Vector autoregressive models, autocovariance and autocorrelation functions, confidence bands, delta method, Phillips curve

JEL Classification System: C13, C32, E31

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1 Introduction

Vector autoregressive (VAR) models are one of the most popular classes of models in applied econometrics. They provide a simple tool for characterising the dynamic interaction of the data, which can be displayed either by their autocovariance and autocorrelation functions or by their impulse response functions. Whereas the latter may be sensitive to the validity of the set of assumptions used to identify particular structural shocks, the former are not, because of their purely descriptive nature. In this regard, McCallum (2001) has advocated the more systematic use of autocovariance and autocorrelation functions for confronting structural models with the data. At the same time, there has been renewed interest in reduced-form analysis of autocovariances and autocorrelations that is only in part driven by dissatisfaction with structural VAR analysis. For instance, covariance and correlation analysis plays an important role in the recent literature on the inflation-output tradeoff and in the evaluation of the fit of dynamic general equilibrium models (see, among others, Fuhrer and Moore (1995), Rotemberg and Woodford (1997), Nelson (1998), Galí and Gertler (1999), Kiley (2000) and Barsky and Kilian (2001)).

Although the computation of the autocovariance and autocorrelation functions of VAR models is straightforward from a technical point of view, there often remains a fundamental shortcoming in applied work. The autocovariance and autocorrelation functions are computed from coefficients of VAR models which are estimated from the data. The former are therefore also estimates and, hence, affected by uncertainty. Often researchers only report point estimates of the autocovariance and autocorrelation functions and do not properly take into account the underlying estimation uncertainty. By contrast, we present an easy to use method for constructing asymptotic confidence bands which allow to assess this uncertainty. Specifically, we employ the delta method and provide simple closed-form formulae for computing the asymptotic covariance matrices of the autocovariance and autocorrelation functions of stable finite-order VAR models. Asymptotic confidence bands are then easily constructed using the standard errors obtained from these matrices.

It is well known that asymptotic confidence bands for autocovariances and autocorrelations of the data can be simply derived under the null hypothesis that the data are generated by a white-noise process. In this case, the autocovariances and autocorrelations are asymptotically normal with the standard errors of the estimated autocorrelations being approximately equal to $1/\sqrt{T}$ (see for example Lütkepohl (1991), Chapter 4.4). Tests based on the estimated autocorrelations are thus very easy to conduct. In applied work, however, often we are not particularly interested in testing the null of zero autocorrelation. In fact, we may not have any specific null in mind. For this reason we are interested in constructing confidence bands centred around the point estimates of autocovariance and autocorrelation functions, as opposed to conducting tests.

The remainder of the paper is organised as follows. In Section 2 we establish the asymptotic distribution of the autocovariance and autocorrelation functions of stable finite-order VAR models and provide closed-form formulae for computing the partial derivatives of the autocovariance and autocorrelation functions with respect to the underlying VAR coefficients which are needed to compute their asymptotic covariance matrices. In Section 3 we illustrate the usefulness of our formulae for empirical work by an application to inflation and output gap data for the United States. Section 4 concludes.

2 The Asymptotic Distribution of the Autocovariance and Autocorrelation Functions of Stable VAR Models

Before stating the asymptotic distribution of the estimated autocovariance and autocorrelation functions in Section 2.3, we briefly review some results on the estimation of stable finite-order VAR models and the computation of their autocovariance and autocorrelation functions. These results, which are provided in Sections 2.1 and 2.2 respectively, are utilised later on when establishing the asymptotic property of interest.[1]

2.1 The Stable VAR(p) Model

Let $\{ y_t : t = 0, \pm 1, \dots \}$ be a sequence of a k-dimensional vector of variables which is generated by an unrestricted vector autoregressive (VAR) process of finite lag order p,

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \qquad t = 0, \pm 1, \dots \tag{1}$$

with $\{ u_t : t = 0, \pm 1, \dots \}$ being serially uncorrelated with mean zero and positive definite covariance matrix Σ_u.[2]

The VAR(p) model (1) is assumed to be stable, i.e.

$$\det(I_k - A_1 z - \cdots - A_p z^p) \neq 0 \quad \text{for} \quad |z| \leq 1,$$

where $|\cdot|$ denotes the modulus operator.

Let $f(y_{-p+1}, \dots, y_0; \beta) \prod_{t=1}^{T} f(y_t \mid y_{t-p}, \dots, y_{t-1}; \beta)$ be the density of a sample $\{ y_t : t = -p+1, \dots, T \}$ generated by the VAR(p) process. Then, for fixed pre-sample values $y_{-p+1}, \dots, y_0$, the conditional maximum-likelihood (ML) estimator for $\beta \in B$ is

$$\hat{\beta}_T = \arg\max_{\beta \in B} \sum_{t=1}^{T} \ln f(y_t \,|\, y_{t-p}, \dots, y_{t-1}; \beta),$$

where

$$\beta = \left[\operatorname{vec}(A_1, \dots, A_p)', \operatorname{vech}(\Sigma_u)' \right]'$$

is the n-dimensional parameter vector of the VAR(p) model with $n = k^2 p + k(k+1)/2$ and $B \subset \mathbf{R}^n$ denotes the feasible parameter space. The vec($\cdot$)-operator stacks the columns of

[1] For reference and further details see Lütkepohl (1991).

[2] In practice, the VAR process in equation (1) may also include deterministic terms, such as an intercept or a linear trend. Here, the intercept and trend coefficients have been normalised to zero in population. Alternatively, if an intercept or a linear trend is included in the estimation, the same data-generating process can be interpreted as deviations from the implicit mean or time trend.

a matrix in a column vector and the vech(·)-operator stacks the elements on and below the principal diagonal of a square matrix (see Lütkepohl (1996), Chapter 1.4).

Under general regularity conditions the ML estimator $\hat{\beta}_T$ converges in probability to the "true" parameter vector β_0,

$$\operatorname*{plim}_{T\to\infty} \hat{\beta}_T = \beta_0,$$

and is jointly asymptotically normal,

$$\sqrt{T}\,(\hat{\beta}_T - \beta_0) \xrightarrow{d} \mathrm{N}\left[0, \Sigma_{\hat{\beta}}(\beta_0)\right],$$

where $\Sigma_{\hat{\beta}}(\beta_0)$ is the asymptotic covariance matrix of $\sqrt{T}(\hat{\beta}_T - \beta_0)$.[3]

2.2 The Autocovariance and Autocorrelation Functions

In order to estimate the autocovariance and autocorrelation functions of the stable VAR(p) model it is convenient to start from its VAR(1) representation,

$$Y_t = A\,Y_{t-1} + U_t, \qquad t = 0, \pm 1, \ldots \tag{2}$$

with

$$Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}, \qquad U_t = \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_k & 0 & \cdots & 0 & 0 \\ 0 & I_k & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_k & 0 \end{bmatrix}, \qquad \Sigma_U = \begin{bmatrix} \Sigma_u & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The (one-sided) autocovariance function implied by the VAR(1) model (2), $\{\Gamma_{Y,h} : h = 0, 1, \ldots\}$ with $\Gamma_{Y,h} = \Gamma_{Y,h}(\beta) = \mathrm{E}[\,Y_t\,Y'_{t-h}\,]$, is obtained in two steps. First, the stacked contemporaneous covariance matrix fulfills the equation

$$\mathrm{vec}(\Gamma_{Y,0}) = \Big(I_{(kp)^2} - A \otimes A\Big)^{-1} \mathrm{vec}(\Sigma_U), \tag{3}$$

where $\otimes$ denotes the Kronecker product (see Lütkepohl (1996), Chapter 1.2). And second, the higher order autocovariance matrices are given recursively by the Yule-Walker equation of the VAR(1) model,

$$\Gamma_{Y,h} = A\,\Gamma_{Y,h-1}, \qquad h = 1, 2, \ldots. \tag{4}$$

[3]Closed-form expressions for the estimator $\hat{\beta}_T$ and the asymptotic covariance matrix $\Sigma_{\hat{\beta}}(\beta_0)$ are available from Chapter 3.4 in Lütkepohl (1991).

Then, because

$$\Gamma_{Y,h} = \begin{bmatrix} \Gamma_{y,h} & \Gamma_{y,h+1} & \cdots & \Gamma_{y,h+p-1} \\ \Gamma_{y,h-1} & \Gamma_{y,h} & \cdots & \Gamma_{y,h+p} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{y,h-p+1} & \Gamma_{y,h-p} & \cdots & \Gamma_{y,h} \end{bmatrix}$$

with $\Gamma_{y,h} = \Gamma'_{y,-h}$, the autocovariance function of the VAR(p) model (1), $\{\, \Gamma_{y,h} : h = 0, \pm 1, \dots \}$ with $\Gamma_{y,h} = \Gamma_{y,h}(\beta) = \mathrm{E}[\, y_t \, y'_{t-h} \,]$, can be easily recovered from the autocovariance function of its VAR(1) representation by means of an appropriately defined $(0, 1)$ selection matrix,

$$\Gamma_{y,h} = J \, \Gamma_{Y,h} \, J' \tag{5}$$

with $J = [\; I_k \; 0 \; \cdots \; 0 \;]$.

Estimating the autocovariance function via the VAR model, i.e., making a parametric assumption regarding the data generating mechanism of $\{\, y_t : t = 0, \pm 1, \dots \}$, rather than estimating it directly from the data using $T^{-1} \sum_t y_t \, y'_{t-h}$, is considered useful since the estimation of higher order autocovariances without a parametric assumption tends to be unreliable in finite samples.

Given the autocovariance function $\{\, \Gamma_{y,h} : h = 0, \pm 1, \dots \}$, the autocorrelation function, $\{\, R_{y,h} : h = 0, \pm 1, \dots \}$ with $R_{y,h} = R_{y,h}(\beta)$, is defined by

$$R_{y,h} = D^{-1} \Gamma_{y,h} \, D^{-1}, \qquad h = 0, \pm 1, \dots , \tag{6}$$

where D is a diagonal matrix with its diagonal elements being the square roots of the diagonal elements of $\Gamma_{y,0}$.

Replacing the unknown parameter vector β with its ML estimate $\hat{\beta}_T$, we obtain the estimated autocovariance and autocorrelation functions $\{\, \hat{\Gamma}_{y,h} : h = 0, \pm 1, \dots \}$ and $\{\, \hat{R}_{y,h} : h = 0, \pm 1, \dots \}$ with $\hat{\Gamma}_{y,h} = \Gamma_{y,h}(\hat{\beta}_T)$ and $\hat{R}_{y,h} = R_{y,h}(\hat{\beta}_T)$ respectively.

2.3 Asymptotic Normality of the Autocovariance and Autocorrelation Functions

The asymptotic distribution of the estimated autocovariance and autocorrelation functions follows immediately from the application of the delta method based on results for continuously differentiable functions of consistently estimated, asymptotically normally distributed parameters (see Serfling (1980), Theorem 3.3.A).

Specifically, recalling the consistency and the asymptotic normality of the ML estimator $\hat{\beta}_T$, the estimators of the autocovariance and autocorrelation functions, $\{\, \hat{\Gamma}_{y,h} : h = 0, \pm 1, \dots \}$ and $\{\, \hat{R}_{y,h} : h = 0, \pm 1, \dots \}$, are (pointwise) asymptotically normal as well:

$$\sqrt{T} \, \Big(\mathrm{vec}(\hat{\Gamma}_{y,h} - \Gamma_{y,h}(\beta_0)) \Big) \;\xrightarrow{d}\; \mathrm{N} \Big[0, \Sigma_{\mathrm{vec}(\hat{\Gamma}_{y,h})}(\beta_0) \Big] , \qquad h = 0, \pm 1, \dots ,$$

where

$$\Sigma_{\mathrm{vec}(\hat{\Gamma}_{y,h})}(\beta_0) = \frac{\partial \mathrm{vec}(\Gamma_{y,h})}{\partial \beta'} \, \Sigma_{\hat{\beta}}(\beta_0) \, \frac{\partial \mathrm{vec}(\Gamma_{y,h})'}{\partial \beta},$$

and

$$\sqrt{T} \left(\mathrm{vec}(\hat{R}_{y,h} - R_{y,h}(\beta_0)) \right) \overset{d}{\longrightarrow} \mathrm{N}\left[0, \Sigma_{\mathrm{vec}(\hat{R}_{y,h})}(\beta_0) \right], \qquad h = 0, \pm 1, \dots ,$$

where

$$\Sigma_{\mathrm{vec}(\hat{R}_{y,h})}(\beta_0) = \frac{\partial \mathrm{vec}(R_{y,h})}{\partial \beta'} \, \Sigma_{\hat{\beta}}(\beta_0) \, \frac{\partial \mathrm{vec}(R_{y,h})'}{\partial \beta},$$

with the partial derivates of the autocovariance and autocorrelation matrices being evaluated at β_0.

In order to obtain the covariance matrices of the asymptotic distributions of the autocovariance and autocorrelation functions we need their partial derivatives with respect to the parameter vector β. These derivatives can be determined by straightforward application of matrix differential calculus.[4]

Due to the symmetry of the autovariance and autocorrelation functions, it suffices to consider the derivatives for $h = 0, 1, \dots$. First, using equation (5) and recursively substituting equation (4), the partial derivatives of the autocovariance matrices of order $h = 0, 1, \dots$ are given by

$$\frac{\partial \mathrm{vec}(\Gamma_{y,h})}{\partial \beta'} = \left(J \otimes J \, A^h \right) \frac{\partial \mathrm{vec}(\Gamma_{Y,0})}{\partial \beta'} + \left(J \, \Gamma_{Y,0}' \otimes J \right) \frac{\partial \mathrm{vec}(A^h)}{\partial \beta'},$$

where, recalling equation (3),

(a) $$\frac{\partial \mathrm{vec}(\Gamma_{Y,0})}{\partial \beta'} = \left(\mathrm{vec}(\Sigma_U)' \otimes I_{(kp)^2} \right) \frac{\partial \mathrm{vec}(\Psi(A)^{-1})}{\partial \mathrm{vec}(\Psi(A))'} \, \frac{\partial \mathrm{vec}(\Psi(A))}{\partial \mathrm{vec}(A \otimes A)'} \times \frac{\partial \mathrm{vec}(A \otimes A)}{\partial \mathrm{vec}(A)'} \, \frac{\partial \mathrm{vec}(A)}{\partial \beta'} + \Psi(A)^{-1} \, \frac{\partial \mathrm{vec}(\Sigma_U)}{\partial \beta'}$$

with

$$\begin{aligned}
\Psi(A) &= I_{(kp)^2} - A \otimes A, \\
\frac{\partial \mathrm{vec}(\Psi(A)^{-1})}{\partial \mathrm{vec}(\Psi(A))'} &= -\left(\Psi(A)'\right)^{-1} \otimes \Psi(A)^{-1}, \\
\frac{\partial \mathrm{vec}(\Psi(A))}{\partial \mathrm{vec}(A \otimes A)'} &= -\, I_{(kp)^4}, \\
\frac{\partial \mathrm{vec}(A \otimes A)}{\partial \mathrm{vec}(A)'} &= (I_{kp} \otimes K_{kp,kp} \otimes I_{kp})\Big(\Big(I_{(kp)^2} \otimes \mathrm{vec}(A)\Big) + \Big(\mathrm{vec}(A) \otimes I_{(kp)^2}\Big)\Big), \\
\frac{\partial \mathrm{vec}(A)}{\partial \beta'} &= \left[\; (I_{kp} \otimes \left[\, 1 \;\; 0_{1,p-1} \, \right]' \otimes I_k) \quad 0_{(kp)^2, k(k+1)/2} \; \right],
\end{aligned}$$

[4] For a comprehensive collection of results on matrix differential calculus see Lütkepohl (1996) or Magnus and Neudecker (1988).

where $K_{kp,kp}$ denotes the $((kp)^2 \times (kp)^2)$-dimensional commutation matrix defined such that $\mathrm{vec}(A) = K_{kp,kp}\,\mathrm{vec}(A')$ (see Lütkepohl (1996), Chapter 1.5), and

$$\frac{\partial \mathrm{vec}(\Sigma_U)}{\partial \beta'} = \Bigg[\; 0_{(kp)^2,k^2p} \quad (I_p \otimes K_{k,p} \otimes I_k) \times \left(\mathrm{vec}\left(\begin{bmatrix} 1 & 0_{1,p-1} \\ 0_{p-1,1} & 0_{p-1,p-1} \end{bmatrix} \right) \otimes D_k \right) \Bigg],$$

where D_k denotes the $(k^2 \times k(k+1)/2)$-dimensional duplication matrix defined such that $\mathrm{vec}(\Sigma_u) = D_k\,\mathrm{vech}(\Sigma_u)$ (see Lütkepohl (1996), Chapter 1.5),

and

(b) $$\frac{\partial \mathrm{vec}(A^h)}{\partial \beta'} = \frac{\partial \mathrm{vec}(A^h)}{\partial \mathrm{vec}(A)'} \, \frac{\partial \mathrm{vec}(A)}{\partial \beta'}$$

with

$$\frac{\partial \mathrm{vec}(A^h)}{\partial \mathrm{vec}(A)'} = \begin{cases} 0_{(kp)^2,(kp)^2}, & h = 0 \\ \sum_{i=0}^{h-1} (A')^{h-1-i} \otimes A^i, & h = 1, 2, \ldots . \end{cases}$$

To establish the partial derivatives under (a) and (b) we repeatedly employed the product rule and the chain rule for matrix-valued functions with vector arguments (see Lütkepohl (1996), Chapter 10).

Second, recalling equation (6), the partial derivatives of the autocorrelation matrices of order $h = 0, 1, \ldots$ are given by

$$\frac{\partial \mathrm{vec}(R_{y,h})}{\partial \beta'} = \left(D^{-1} \otimes D^{-1} \right) \frac{\partial \mathrm{vec}(\Gamma_{y,h})}{\partial \beta'} + \left(\left(I_k \otimes D^{-1}\Gamma_{y,h} \right) + \left(D^{-1}\Gamma'_{y,h} \otimes I_k \right) \right) \frac{\partial \mathrm{vec}(D^{-1})}{\partial \beta'},$$

where

$$\frac{\partial \mathrm{vec}(D^{-1})}{\partial \beta'} = \frac{\partial \mathrm{vec}(D^{-1})}{\partial \mathrm{vec}(D)'} \, \frac{\partial \mathrm{vec}(D)}{\partial \mathrm{vec}(\Gamma_{y,0})'} \, \frac{\partial \mathrm{vec}(\Gamma_{y,0})}{\partial \beta'}$$

with

$$\frac{\partial \mathrm{vec}(D^{-1})}{\partial \mathrm{vec}(D)'} = -\,D^{-1} \otimes D^{-1},$$

$$\frac{\partial \mathrm{vec}(D)}{\partial \mathrm{vec}(\Gamma_{y,0})'} = 0.5\,\mathrm{diag}\left(\mathrm{vec}(D^{-1}) \right),$$

where the $\mathrm{diag}(\,\cdot\,)$-operator constructs a diagonal matrix from the elements of the vector argument (see Lütkepohl (1996), Chapter 1.4), and where the partial derivatives of the autocovariance matrices of order $h = 0, 1, \ldots$ are given above.

Using the closed-form formulae for computing the partial derivatives of the autocovariance and autocorrelation matrices, the asymptotic covariance matrices of the estimated autocovariance and autocorrelation functions can be calculated by replacing the unknown parameter vector β with its ML estimate $\hat{\beta}_T$ and using an appropriate estimate of the asymptotic covariance matrix of the latter. Asymptotic confidence bands are then easily constructed from the estimated asymptotic standard errors of the autocovariance and autocorrelation functions, that is the square roots of the elements on the principal diagonal of their estimated asymptotic covariance matrices.

It is important to note that the results regarding the (pointwise) asymptotic normality of the estimated autocovariance and autocorrelation functions only hold if the associated asymptotic covariance matrices are non-singular. Obviously, these matrices may be singular if some of the unknown parameters β are zero and/or restricted with $\Sigma_{\hat{\beta}}$ being singular as a result. However, even if $\Sigma_{\hat{\beta}}$ is non-singular, the asymptotic covariance matrices of the estimated autocovariance and autocorrelation functions can be singular because their partial derivatives are not necessarily non-zero in the entire parameter space. In such situations the estimated autocovariance and autocorrelation functions will have a degenerate limiting distribution and the use of asymptotic confidence bands that are constructed from the asymptotic covariance matrices by replacing the unknown parameters by their usual estimates may not be appropriate.[5]

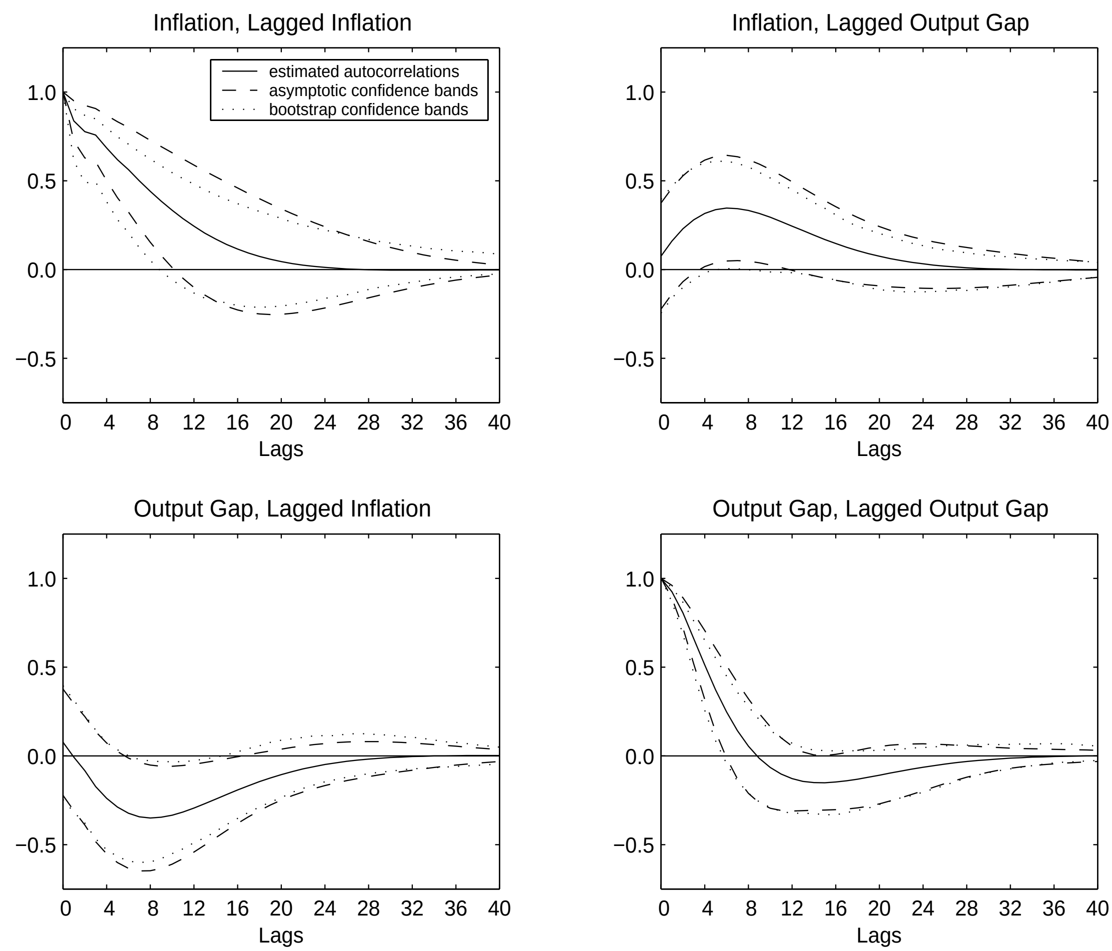
3 An Illustrative Example

In a widely-quoted paper, Fuhrer and Moore (1995) investigated the dynamic characteristics of the inflation and output gap processes for the U.S. economy by means of the estimated autocorrelation function of a finite-order VAR model. They pursued two objectives. First, albeit without assessing the underlying estimation uncertainty, they used the autocorrelation function as a descriptive device to investigate the lead-lag relationship between inflation and the output gap which traditionally forms the empirical basis of structural models of the short-run Phillips-curve tradeoff. Second, they used the estimated autocorrelation function as a benchmark against which the ability of alternative structural models to explain the observed degree of inflation persistence was evaluated.

In our example, we focus on the first of the two objectives and revisit the pattern of inflation and output gap dynamics in the United States. Building on Fuhrer and Moore (1995), we estimate the autocorrelation function of a VAR model fitted to data on the quarter-on-quarter change in the log-level of the GDP deflator and the log-level of real GDP in deviation from a linear trend. The time series span the period from the first quarter of

[5] See Benkwitz, Lütkepohl and Neumann (2000) for a study of the potentially dramatic consequences for conducting inference on the impulse response function computed from a simple univariate AR(1) model if the limiting distribution is degenerate.

Figure: Inflation and Output Gap Dynamics in the United States



1965 to the third quarter of 2001. We choose a lag order of 3 employing a standard lag selection procedure based on the HQ criterion. The point estimate of the autoregressive polynomial implies a dominant pair of complex eigenvalues with modulus 0.848, suggesting that the VAR model is stable and, hence, that the associated autocorrelation function is well-defined.[6]

The figure above shows the point estimates (solid line) and the asymptotic 95% confidence bands (dashed lines) for the autocorrelation function implied by the estimated VAR model. The panels on the diagonal of the figure pertain to the autocorrelations of inflation and the output gap, the off-diagonal panels to their cross-autocorrelations.[7] The autocor-

[6] This finding is broadly supported by the results of augmented Dickey-Fuller tests for unit roots in the two time series. The values of the relevant t-statistics for the log of real GDP and the quarter-on-quarter change in the log of the GDP deflator are -3.515 and -2.577, which are significant at the 5% and 10% levels respectively (see Davidson and MacKinnon (1993), Table 20.1).

[7] Apparently, since the elements on the diagonal of the autocorrelation matrix of order $h = 0$ are

relations are indicative of a quite substantial degree of persistence in the inflation rate, while the output gap is somewhat less persistent. The cross-autocorrelations in the upper right-hand panel show that the output gap leads the inflation rate by about six quarters, thereby suggesting the existence of a short-run Phillips-curve tradeoff. This tradeoff proves to be statistically significant, as revealed by the asymptotic confidence bands. By contrast, the lower left-hand panel displays that the lagged inflation rate is negatively correlated with the output gap.[8]

A comparison of the asymptotic 95% confidence bands with confidence bands based on the 2.5% and 97.5% percentiles of the empirical distribution of bootstrap estimates of the autocorrelations (dotted lines) reveals that the asymptotic confidence bands are rather close to the bootstrap confidence bands.[9] In particular, the bootstrap-based confidence bands tend to confirm both the existence of a short-run Phillips-curve tradeoff and the negative correlation of lagged inflation and the output gap. As to the autocorrelations of inflation, it is worth noting that the bootstrap confidence bands indicate a somewhat lower degree of inflation persistence, a possible reason being that the bootstrap confidence bands attempt to allow for asymmetry in the distribution of the higher-order autocorrelations while the asymptotic confidence bands are symmetrically centred around the point estimates of the autocorrelations.

4 Conclusion

In applied work, researchers often report the autocovariance or autocorrelation functions computed from an estimated VAR model in order to characterise the dynamics of the data. However, since the autocovariance and autocorrelation functions depend on the estimated parameters of the underlying VAR model, the former are estimates themselves and, hence, their proper interpretation requires to take the estimation uncertainty into account. As a means to accomplish this, we provide simple closed-form formulae for constructing asymptotic confidence bands for the estimated autocovariance and autocorrelation functions of stable finite-order VAR models. We furthermore show by an illustrative application to inflation and output gap data for the U.S. economy that plotting the point estimates of the autocorrelation function together with their estimated asymtotic confidence bands offers a useful tool for assessing the estimation uncertainty involved.

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equal to one by construction, their partial derivatives are zero and the limiting distribution has a singular covariance matrix. This problem, however, can be easily avoided by pre-multiplying the object of interest, $\text{vec}(\hat{R}_{y,0} - R_{y,0})$, with an appropriately defined $(0, 1)$ selection matrix such that a non-degenerate limiting distribution is obtained.

[8] Results for VAR models including deviations of real GDP from the CBO measure of potential output rather than trend output and/or the Federal Funds Rate confirm these findings.

[9] The bootstrap confidence bands are generated from 2,000 bootstrap versions of $\hat{\beta}_T^*$, conditional on the estimated parameters of the VAR(3) model, $\hat{\beta}_T$, the estimation residuals, $\hat{u}_t$, and the pre-sample values y_{-2}, y_{-1}, y_0, using standard techniques (see, e.g., Runkle (1987)).

We notice that there are several interesting extensions which are beyond the scope of this paper. First, we have not explored how the proposed method for constructing asymptotic confidence bands for estimated autocovariance and autocorrelation functions behaves in small samples. Second, we have not systematically investigated how the asymptotic confidence bands perform relative to confidence bands based on bootstrap techniques which have gained increased popularity in applied work. For example, while the illustrative example indicates that the asymptotic method may give results quite similar to those obtained by standard bootstrap techniques, we recognise that the use of more advanced bootstrap methods, such as the bias-corrected bootstrap technique proposed by Kilian (1998), may outperform the asymptotic method. The study of these issues should be an interesting topic for future research.

References

Barsky R, Kilian L (2001) Do we really know that oil caused the great stagflation? A monetary alternative. In: Bernanke BS, Rotemberg JJ (eds.) NBER Macroeconomics Annual 2001 Vol. 16. MIT Press, Cambridge, MA, pp. 137-183

Benkwitz A, Lütkepohl H, Neumann MH (2000) Problems related to confidence intervals for impulse responses of autoregressive processes. Econometric Reviews 19:69-103

Davidson R, MacKinnon JG (1993) Estimation and inference in econometrics. Oxford University Press, New York

Fuhrer JC, Moore GR (1995) Inflation persistence. Quarterly Journal of Economics 110:127-160

Galí J, Gertler M (1999) Inflation dynamics: A structural econometric analysis. Journal of Monetary Economics 44:195-222

Kiley MT (2000) Endogenous price stickiness and business cycle persistence. Journal of Money, Credit and Banking 32:28-53

Kilian L (1998) Small-sample confidence intervals for impulse response functions. Review of Economics and Statistics 80:218-230

Lütkepohl H (1991) Introduction to multiple time series analysis. Springer-Verlag, Berlin

Lütkepohl H (1996) Handbook of matrices. John Wiley, Chichester

Magnus JR, Neudecker H (1988) Matrix differential calculus with applications in statistics and econometrics. John Wiley, Chichester

McCallum BT (2001) Analysis of the monetary transmission mechanism: Methodological issues. In: Deutsche Bundesbank (ed.) The monetary transmission process: Recent developments and lessons for Europe. Palgrave, Basingstoke, pp. 11-43

Nelson E (1998) Sluggish inflation and optimizing models of the business cycle. Journal of Monetary Economics 42:303-322

Rotemberg JJ, Woodford M (1997) An optimization-based econometric model for the evaluation of monetary policy. In: Bernanke BS, Rotemberg JJ (eds.) NBER Macroeconomics Annual 1997 Vol. 12. MIT Press, Cambridge, MA, pp. 297-346

Runkle DE (1987) Vector autoregressions and reality. Journal of Business and Economic Statistics 5:437-442

Serfling RJ (1980) Approximation theorems of mathematical statistics. John Wiley, New York